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Existence of Periodic Solutions of the Navier–Stokes Equations

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1. INTRODUCTION AND SUMMARY

The Navier–Stokes equations describe the motion of viscous incompressible fluids. We assume that the incompressible fluids are homogeneous and that the density ρ and kinematic viscosity ν of the fluids are constants independent of x and t. Then the Navier–Stokes equations are

$$\rho(u_t + u \cdot \nabla u) - \nu \Delta u = f - \nabla p,$$
$$\operatorname{div} u = 0.$$

The present paper is concerned with the unique existence of strong periodic solutions of the Navier–Stokes equations with $\rho = 1$ and $\nu = 1$,

$$u_t - \Delta u + u \cdot \nabla u = f - \nabla p, \qquad x \in \Omega, t \in R^1, \tag{1.1}$$

$$\operatorname{div} u = 0, \qquad x \in \Omega, t \in R^1, \tag{1.2}$$

$$u\,|_{\partial\Omega} = 0, \qquad t \in R^1, \tag{1.3}$$

where Ω is a bounded domain in the N-dimensional Euclidean space R^N with smooth boundary $\partial\Omega$, the vector-valued function $u = (u^1(x,t), \dots, u^N(x,t))$ is the unknown velocity field, the scalar function

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$p = p(x,t)$ is the unknown pressure, the vector-valued function $f = (f^1(x,t),\dots,f^N(x,t))$ is the given external force, and the notation

$$u \cdot \nabla v = \sum_{i=1}^{N} u^i \frac{\partial v}{\partial x_i}, \qquad \operatorname{div} u = \sum_{i=1}^{N} \frac{\partial u^i}{\partial x_i}$$

is used for the vector-valued functions u and v.

Equation (1.1) can be considered as the nondimensionalized form of the Navier–Stokes equations with $Re = 1$ (Re: Reynolds number). The unique existence of strong periodic solutions of the Navier–Stokes equations with $Re > 1$ can be studied similarly. The scaling in the smallness assumption on f introduced later will be affected by the choice of Re.

The problem we consider is as follows. Let the given external force f be periodic in t with some period ω. Then we try to prove the existence and uniqueness of periodic strong solutions u of the Navier–Stokes equations (1.1)–(1.3) with the same period ω:

$$u(x,t+\omega) = u(x,t), \qquad x \in \Omega,\ t \in R^1. \tag{1.4}$$

As is well known, there are a number of papers concerning the initial value problem for the Navier–Stokes equations, that is, (1.1)–(1.3) together with

$$u(x,0) = a(x), \qquad x \in \Omega \tag{1.5}$$

(in particular, concerning the existence and the uniqueness, see Fujita and T. Kato [2], Giga and Miyakawa [4], Hopf [5], Ito [6], Ladyzhenskaia [9], Lions [10], Lions and Prodi [11], Masuda [12], Serrin [13], Temam [15], Wahl [16], etc.). On the other hand, for the periodicity problem (1.1)–(1.4), Kaniel and Shinbrot [7] have shown a reproductive property of the Navier–Stokes equations for $N = 3$ when the external force f is sufficiently small, i.e., $\sup_{0 \le t < \infty} \|f(t)\|_{L_2(\Omega)}$ is small. For 2-dimensions Takeshita [14] also obtained the reproductive property under no smallness assumptions on f. Recently, Kozono and Nakao [8] (preprint) have proved the existence of a periodic strong solution for any dimension $N \ge 3$ under the assumption that $\sup_{0 \le t \le \omega} \|f(t)\|_{L_r(\Omega)}$ $(r > N/2)$ is sufficiently small by considering the integral equation in a Banach space. They have also studied the problem in some unbounded domains.

In this paper we shall show the unique existence of strong solutions of the periodicity problem (1.1)–(1.4) for $N = 3$ and $N = 4$. We would like to emphasize that we shall prove our uniqueness and existence theorems under the *critical* smallness assumption, i.e.,

$$\sup_{0 \le t \le \omega} \|f(t)\|_{L_{N/2}(\Omega)} \text{ is sufficiently small.}$$

For this purpose, fairly precise energy estimates will be required, and our result is on the grounds that we are able to apply L_2-theory together with the fractional power of the Stokes operator. Since the Stokes operator (denoted by A) is strictly positive self-adjoint in the Hilbert space, the fractional power of A is defined by its spectral resolution. However, when we consider the problem in the Hilbert space, we need to place the restriction on the space dimension N because of the nonlinear term appearing in the equations. We shall give the definition of the Stokes operator later.

To describe our theorems accurately, we introduce some basic function spaces and notion. We define

$$C_{0,\sigma}^{\infty} \equiv \{\varphi \in C_0^{\infty}(\Omega);\, \operatorname{div} \varphi = 0\}.$$

In addition, we define H_σ as the closure of $C_{0,\sigma}^{\infty}$ in $L_2(\Omega)$, and $H_{0,\sigma}^1$ as the closure of $C_{0,\sigma}^{\infty}$ in $H^1(\Omega)$. Throughout this paper, $L_2(\Omega)$ represents the Hilbert space equipped with the inner product

$$(u, v) = \sum_{i=1}^{N} \int_{\Omega} u^i v^i \, dx.$$

We denote the $L_2(\Omega)$-norm by $\|\cdot\|$. $H^m(\Omega)$ is the Sobolev space of vector-valued functions which are in $L_2(\Omega)$ together with their derivatives up to order m. $H_0^m(\Omega)$ is the completion of the set $C_0^{\infty}(\Omega)$ in $H^m(\Omega)$.

Let P be the orthogonal projection from $L_2(\Omega)$ onto H_σ. By the Stokes operator A we denote the Friedrichs extension of the symmetric operator $-P\Delta$ in H_σ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega) \cap H_\sigma$. It is well known that $D(A^{1/2}) = H_{0,\sigma}^1$. Then, the periodicity problem (1.1)–(1.4) is formulated as an ordinary differential equation in the Hilbert space H_σ (see Fujita and T. Kato [2]),

$$\frac{du}{dt} + Au + Pu \cdot \nabla u = Pf(t), \qquad t \in R^1, \tag{1.6}$$

$$u(t + \omega) = u(t), \qquad t \in R^1, \tag{1.7}$$

by the orthogonal decomposition

$$L_2(\Omega) = H_\sigma \oplus \{\nabla p;\, p \in H^1(\Omega)\}$$

(see Fujiwara and Morimoto [3], Ladyzhenskaia [9]).

Next, let us introduce some function spaces consisting of ω-periodic functions. Let X be a Banach space. We denote by $C^k(\omega; X)$ (k: nonnegative integers) the set of X-valued ω-periodic functions on R^1 with continuous derivatives up to order k. Then let us define the norm

$$\|f\|_{C^k(\omega;X)} = \sup_{0\le t\le\omega}\left\{\sum_{i=0}^{k}\|D_t^i f(t)\|_X\right\}.$$

We denote by $L_p(\omega; X)$ $(1 \le p \le \infty)$ the set of ω-periodic X-valued measurable functions f on R^1 such that

$$\|f\|_{L_p(\omega;X)} = \left(\int_0^\omega \|f(t)\|_X^p\,dt\right)^{1/p} < +\infty \qquad (1 \le p < \infty),$$

$$\|f\|_{L_\infty(\omega;X)} = \sup_{0\le t\le\omega}\|f(t)\|_X < +\infty.$$

We denote by $W^{k,p}(\omega; X)$ the set of functions f which belong to $L_p(\omega; X)$ together with their derivatives up to order k, and in particular we write $H^k(\omega; X) = W^{k,2}(\omega; X)$ when X is a Hilbert space.

To prove our theorems, we shall use the remarkable following proposition on estimates of the nonlinear term. (We state the proposition in the Hilbert space.)

PROPOSITION 1.1 (Giga and Miyakawa [3, p. 270]). *If* $0 \le \delta < 1/2 + N/4$, *the following estimate is valid with a constant* $C_1 = C_1(\delta, \theta, \rho)$,

$$\|A^{-\delta}Pu\cdot\nabla v\| \le C_1\|A^\theta u\|\,\|A^\rho v\| \quad \text{for any } u \in D(A^\theta) \text{ and } v \in D(A^\rho), \tag{1.8}$$

with $\delta + \theta + \rho \ge N/4 + 1/2$, $\rho + \delta > 1/2$, *and* $\theta, \rho > 0$.

Now, our existence and uniqueness theorems are as follows: Ω is a bounded domain in R^N $(N = 3, 4)$ with smooth boundary.

THEOREM 1.1 (Existence). *Let* $f \in H^1(\omega; H_\sigma)$ $(\omega > 0)$. *Then there exists a constant* $K_0 = K_0(N) > 0$ *such that if*

$$M \equiv \sup_{0\le t\le\omega}\|f\|_{L_{N/2}(\Omega)} \le K_0, \tag{1.9}$$

the problem (1.6)–(1.7) *has an* ω*-periodic strong solution* $u(t)$ *satisfying*

$$u \in H^2(\omega; H_\sigma) \cap H^1(\omega; D(A)) \cap L_\infty(\omega; D(A)) \cap W^{1,\infty}(\omega; H^1_{0,\sigma}).$$

THEOREM 1.2 (Uniqueness). *The solution of* (1.6)–(1.7) *given in Theorem* 1.1 *is unique*.

In Section 2, we consider the Sobolev inequality (see [1]),

$$\|u\|_{L_r(\Omega)} \le C_2 \|u\|_{H^\beta}, \qquad \text{if } \frac{1}{r} \ge \frac{1}{2} - \frac{\beta}{N} > 0, \tag{1.10}$$

and the inequality due to Giga and Miyakawa [4],

$$\|u\|_{L_r(\Omega)} \le C_3 \|A^\gamma u\|, \qquad \text{if } \frac{1}{r} \ge \frac{1}{2} - \frac{2\gamma}{N} > 0. \tag{1.11}$$

Here, we note that if $r = N$ in (1.11) it follows

$$\|u\|_{L_N(\Omega)} \le C_3 \|A^\gamma u\| \qquad \text{with } \gamma = N/4 - 1/2. \tag{1.12}$$

Further, we shall establish the uniform boundedness of the norm $\|A^\gamma u_n(t)\|$ for approximate solutions u_n of the problem (1.6)–(1.7). In Section 3, we shall derive estimates of derivatives of higher order. For this purpose the boundedness obtained in Section 2 plays an important role. In Section 4, by standard compactness arguments we shall show the convergence of the approximate solutions, and hence we shall obtain the proofs of Theorems 1.1–1.2.

2. APPROXIMATE SOLUTIONS

Firstly, we shall show the existence of approximate solutions of (1.6)–(1.7) under the assumptions in Theorem 1.1. Let $w_i\,(i = 1, 2, \dots)$ be the completely orthonormal system in H_σ consisting of the eigenfunctions of the Stokes operator A. We consider the system of ordinary differential equations,

$$(u_{nt} + Au_n + Pu_n \cdot \nabla u_n, w_i) = (f, w_i) \qquad (i = 1, 2, \dots, n), \tag{2.1}$$

$$u_n(t + \omega) = u_n(t), \qquad \text{where } u_n(t) = \sum_{i=1}^{n} c_{in}(t) w_i. \tag{2.2}$$

Let W_n be the subspace of H_σ spanned by $w_1, w_2, \dots, w_n$. It is well known that for any $v_n(t) = \sum_{i=1}^n d_{in}(t) w_i \in C^1(\omega; W_n)$ there exists a unique ω-periodic solution $u_n(t) = \sum_{i=1}^n c_{in}(t) w_i \in C^1(\omega; W_n)$ of the linear equation

$$(u_{nt} + Au_n, w_i) = (f - Pv_n \cdot \nabla v_n, w_i), \qquad i = 1, 2, \dots, n. \tag{2.3}$$

Moreover, we can see that the mapping $F : v_n \to u_n$ is continuous and compact in $C^1(\omega; W_n)$. Thereby, we shall prove the existence of the solution of (2.1)–(2.2) by applying the Leray–Schauder fixed point theorem. To apply the fixed point theorem it is sufficient to show the boundedness

$$\sup_{0 \le t \le \omega} \|u_n(t)\| \le C \tag{2.4}$$

for all possible solutions of (2.1)–(2.2) replaced by $\lambda P u_n \cdot \nabla u_n$ $(0 \le \lambda \le 1)$ instead of nonlinear term $P u_n \cdot \nabla u_n$, where C is a constant independent of λ.

In fact, multiplying (2.1) by $c_{in}(t)$ and summing up over i we see

$$(u_{nt} + A u_n, u_n) = (f - \lambda P u_n \cdot \nabla u_n, u_n).$$

Hence, from $\operatorname{div} u_n = 0$ and (1.12), we get $(u_n \cdot \nabla u_n, u_n) = 0$, and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_n\|^2 + \|\nabla u_n\|^2 &= (f, u_n) \\ &\le \|f\|_{L_{2N/(N+2)}(\Omega)} \|u_n\|_{L_{2N/(N-2)}(\Omega)} \\ &\le C_3 C(N) \|f\|_{L_{N/2}(\Omega)} \|A^{1/2} u_n\|, \end{aligned} \tag{2.5}$$

where $C(N) \equiv |\Omega|^{(N-2)/2N}$ and $|\Omega| \equiv$ the volume of Ω. Therefore, we get

$$\frac{d}{dt}\|u_n\|^2 + \|\nabla u_n\|^2 \le C_3^2 C(N)^2 M^2, \tag{2.6}$$

where M is defined as (1.9) in Theorem 1.1. Furthermore, considering the periodicity of u_n and integrating (2.6) over $[0, \omega]$ we get

$$\int_0^\omega \|\nabla u_n\|^2 \, dt \le C_3^2 C(N)^2 M^2 \omega. \tag{2.7}$$

By using the spectral representation $A = \int_\mu^\infty \lambda \, dE_\lambda$ ($\mu > 0$: the smallest eigenvalue of A), we obtain the inequality

$$\|A^\alpha u_n\| \le \mu^{\alpha - \beta} \|A^\beta u_n\| \qquad (0 \le \alpha \le \beta). \tag{2.8}$$

Hence, from (2.7) there exists $t^* \in [0, \omega]$ such that

$$\|u_n(t^*)\|^2 \le \mu^{-1} \|\nabla u_n(t^*)\|^2 \le C_3^2 C(N)^2 \mu^{-1} M^2. \tag{2.9}$$

Integrating (2.6) again from t^* to $t + \omega$ ($t \in [0, \omega]$) we can see

$$\sup_{0 \le t \le \omega} \|u_n(t)\|^2 \le 2C_3^2 C(N)^2 M^2 \omega + C_3^2 C(N)^2 \mu^{-1} M^2, \quad (2.10)$$

where the right-hand side of (2.10) is independent of λ and n. Thus, we have proved the existence of the solution $u_n \in C^1(\omega; W_n)$ of (2.1)–(2.2).

Before starting to prove our lemma on the uniform boundedness of $\|A^\gamma u_n(t)\|$ ($\gamma = N/4 - 1/2$), we note that we can choose the basis $\{w_i; i = 1, 2, \dots\}$ such that the eigenfunctions w_i of A are also eigenfunctions of A^γ and that we can write

$$Aw_i = \mu_i w_i, \qquad A^\gamma w_i = \mu_i^\gamma w_i \quad \text{with the eigenvalue } \mu_i \text{ of } A. \quad (2.11)$$

Next we shall show the following lemma with an idea.

LEMMA 2.1. *Let $u_n(t)$ be the solution of* (2.1)–(2.2) *given above. Suppose that*

$$M < K_0 \qquad \text{with } K_0 = \min\{K^{-2}, 1\}, \quad (2.12)$$

and

$$K \equiv C(N)^{-1} \mu^{-\gamma} + C_1 C_3 C(N) \mu^{\gamma - 1/2}.$$

Then we have

$$\|A^\gamma u_n(t)\| \le C_3 C(N) \mu^{\gamma - 1/2} M^{1/2} \qquad \text{for any } t \in (-\infty, +\infty). \quad (2.13)$$

Proof. Considering (2.1)–(2.2) and (2.11) we see

$$\left(u_{nt} + Au_n, A^{2\gamma} u_n\right) = \left(f - Pu_n \cdot \nabla u_n, A^{2\gamma} u_n\right),$$

and further

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|A^\gamma u_n\|^2 &+ \|A^{(1+2\gamma)/2} u_n\|^2 \\ &\le \|f\|_{L_{N/2}(\Omega)} \|A^{2\gamma} u_n\|_{L_{N/(N-2)}(\Omega)} \\ &\quad + \left|\left(A^{(2\gamma-1)/2} Pu_n \cdot \nabla u_n, A^{(1+2\gamma)/2} u_n\right)\right| \\ &\le C_3 M \|A^{(1+2\gamma)/2} u_n\| + C_1 \|A^\gamma u_n\| \, \|A^{(1+2\gamma)/2} u_n\|^2, \quad (2.14) \end{aligned}$$

where we applied (1.8) in Proposition 1.1 and (1.11). By (2.8) and (2.9) we see

$$\|A^\gamma u_n(t^*)\| \le \mu^{\gamma - 1/2} \|\nabla u_n(t^*)\| \le C_3 C(N) \mu^{\gamma - 1/2} M,$$

and hence we get

$$\|A^\gamma u_n(t)\| < C_3 C(N)\mu^{\gamma-1/2}M^{1/2} \qquad \text{at } t = t^*$$

by assuming that $M < 1$. Thus, we set

$$T^* = \sup\{T \mid \|A^\gamma u_n(t)\| \le C_3 C(N)\mu^{\gamma-1/2}M^{1/2} \text{ for any } t \in [t^*, T)\}.$$

Here, we take notice of the order of M. Then, we get $T^* = \infty$. In fact, if T^* $(t^* < T^*)$ is finite it should follow that

$$\|A^\gamma u_n(t)\| \le C_3 C(N)\mu^{\gamma-1/2}M^{1/2} \qquad \text{for any } t \in [t^*, T^*),$$

and

$$\|A^\gamma u_n(T^*)\| = C_3 C(N)\mu^{\gamma-1/2}M^{1/2}. \tag{2.15}$$

Therefore, for such a value $t = T^*$, the estimates of the right-hand side of (2.14) are

$$\begin{aligned} &C_3 M\|A^{(1+2\gamma)/2}u_n(t)\| \\ &\qquad \le C(N)^{-1}\mu^{1/2-\gamma}\|A^\gamma u_n(t)\|\, M^{1/2}\, \|A^{(1+2\gamma)/2}u_n(t)\|, \end{aligned}$$

and

$$\begin{aligned} &C_1\|A^\gamma u_n(t)\|\,\|A^{(1+2\gamma)/2}u_n(t)\|^2 \\ &\qquad \le C_1 C_3 C(N)\mu^{\gamma-1/2}M^{1/2}\|A^{(1+2\gamma)/2}u_n(t)\|^2, \end{aligned}$$

where (2.15) was used. Hence, the estimate (2.14) implies

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|A^\gamma u_n(t)\|^2 &+ \|A^{(1+2\gamma)/2}u_n(t)\|^2 \\ &\le KM^{1/2}\|A^{(1+2\gamma)/2}u_n(t)\|^2 \end{aligned} \tag{2.16}$$

with K defined as (2.12), since the inequality

$$\|A^\gamma u_n\| \le \mu^{-1/2}\|A^{(1+2\gamma)/2}u_n\|$$

is satisfied. Accordingly, from the assumption of this lemma it follows that

$$\frac{d}{dt}\|A^\gamma u_n(t)\|^2 < 0 \qquad \text{at } t = T^*. \tag{2.17}$$

Thus, in a neighborhood of $t = T^*$ it follows

$$\|A^\gamma u_n(t)\| \le C_3 C(N)\mu^{\gamma-1/2}M^{1/2} \qquad \text{for any } t \in [T^*, T^* + \delta),$$

which implies $T^* = \infty$. Moreover, this gives

$$\|A^\gamma u_n(t)\| \le C_3 C(N)\mu^{\gamma-1/2}M^{1/2} \qquad \text{for any } t \in (-\infty, +\infty),$$

because of the periodicity of $u_n(t)$. Consequently, the proof of Lemma 2.1 is complete.

3. ESTIMATES OF DERIVATIVES OF HIGHER ORDER

To show the convergence of the approximate solutions we shall derive estimates of derivatives of higher order. Firstly, we recall Lemma 2.1, namely, that if M is sufficiently small the approximate solutions satisfy

$$\sup_t \|A^\gamma u_n(t)\| \le C(M) \qquad \text{with } \gamma = N/4 - 1/2, \tag{3.1}$$

where $C(M)$ denotes a constant depending on M and independent of n. Moreover, we see easily that if M is sufficiently small then $C(M) < C$ for any positive constant C. Hereafter, we shall use the fact.

LEMMA 3.1. *Let $u_n(t)$ be the solutions of* (2.1)–(2.2) *given above. Set*

$$M_0 \equiv \left(\int_0^\omega \|f\|^2\,dt\right)^{1/2}, \qquad M_1 \equiv \left(\int_0^\omega \|f_t\|^2\,dt\right)^{1/2}.$$

Then, we have

$$\sup_t \|\nabla u_n(t)\| \le C(M_0, M),$$
$$\sup_t \|u_{nt}(t)\| \le C(M_0, M_1, M),$$

where $C(M_0, M)$ and $C(M_0, M_1, M)$ denote constants depending on M_i and M $(i = 0, 1)$ and independent of n.

Proof. From (2.1)–(2.2), similarly, we see

$$(u_{nt} + Au_n, Au_n) = (f - Pu_n \cdot \nabla u_n, Au_n),$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u_n\|^2 + \|Au_n\|^2 \\
&\qquad \le \|f\|\,\|Au_n\| + C_1\|A^\gamma u_n\|\,\|Au_n\|^2 \\
&\qquad \le \|f\|\,\|Au_n\| + C_1C(M)\|Au_n\|^2, \qquad (3.2)
\end{aligned}$$

where we used (3.1) and (1.8) in Proposition 1.1. By integrating (3.2) over $[0, \omega]$ we get

$$\int_0^\omega \|Au_n\|^2\,dt \le M_0\left(\int_0^\omega \|Au_n\|^2\,dt\right)^{1/2} + C_1C(M)\int_0^\omega \|Au_n\|^2\,dt,$$

because of the periodicity of $\nabla u_n(t)$. Seeing that $C_1C(M) < 1$ we obtain

$$\int_0^\omega \|Au_n\|^2\,dt \le C(M_0, M). \qquad (3.3)$$

By (3.3) there exists $t^* \in [0, \omega]$ such that

$$\|\nabla u_n(t^*)\|^2 \le \mu^{-1}\|Au_n(t^*)\|^2 \le \mu^{-1}\frac{C(M_0, M)}{\omega},$$

and by integrating (3.2) from t^* to $t + \omega$ $(t \in [0, \omega])$ we have easily

$$\sup_t \|\nabla u_n(t)\| \le C(M_0, M), \qquad (3.4)$$

where $C(M_0, M)$ is independent of n.

From Eqs. (2.1)–(2.2) again we see

$$(u_{nt} + Au_n, u_{nt}) = (f - Pu_n\cdot\nabla u_n, u_{nt}),$$

and

$$\begin{aligned}
&\|u_{nt}\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u_n\|^2 \\
&\qquad \le \tfrac{1}{2}\|f\|^2 + \tfrac{1}{2}\|u_{nt}\|^2 + \|u_n\|_{L_N(\Omega)}\|\nabla u_n\|\,\|u_{nt}\|_{L_{2N/(N-2)}(\Omega)}, \\
&\qquad \le \tfrac{1}{2}\|f\|^2 + \tfrac{1}{2}\|u_{nt}\|^2 + C_2C_3\|A^\gamma u_n\|\,\|\nabla u_n\|\,\|\nabla u_{nt}\|, \qquad (3.5)
\end{aligned}$$

where we applied the Sobolev inequality (1.10) and (1.12). In the same way, by integrating (3.5) over $[0, \omega]$ we see

$$\int_0^\omega \|u_{nt}\|^2\, dt \le M_0^2 + C(M_0, M)\int_0^\omega \|\nabla u_{nt}\|\, dt. \tag{3.6}$$

Moreover, differentiating Eq. (2.1) in t, multiplying by the derivative $c_{in}'(t)$ defined as (2.2), and summing up over i, we get

$$\begin{aligned}(u_{ntt} &+ Au_{nt}, u_{nt})\\ &= (f_t - Pu_{nt}\cdot\nabla u_n - Pu_n\cdot\nabla u_{nt}, u_{nt}).\end{aligned} \tag{3.7}$$

Noticing that $\operatorname{div} u_n = 0$, and the interpolation inequality

$$\begin{gathered}\|A^\sigma v\| \le C_4\|A^\alpha v\|^\lambda\,\|A^\beta v\|^{1-\lambda} \qquad \big(v \in D(A^\beta)\big),\\ \text{for } \sigma = \alpha\lambda + \beta(1-\lambda),\ 0 < \alpha < \sigma < \beta\ (\lambda \ge 0),\end{gathered} \tag{3.8}$$

we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\|u_{nt}\|^2 + \|\nabla u_{nt}\|^2\\ &\le \mu^{-1/2}\|f_t\|\,\|\nabla u_{nt}\| + C_2C_3\|A^\gamma u_{nt}\|\,\|\nabla u_n\|\,\|\nabla u_{nt}\|\\ &\le \frac{1}{2\mu}\|f_t\|^2 + \frac{1}{2}\|\nabla u_{nt}\|^2 + C_2C_3C_4\|\nabla u_n\|\,\|u_{nt}\|^{1-2\gamma}\|\nabla u_{nt}\|^{1+2\gamma}.\end{aligned} \tag{3.9}$$

In a similar way, we see

$$\begin{aligned}\int_0^\omega &\|\nabla u_{nt}\|^2\, dt\\ &\le \mu^{-1}M_1^2 + 2C_2C_3C_4\int_0^\omega \|\nabla u_n\|\,\|u_{nt}\|^{1-2\gamma}\|\nabla u_{nt}\|^{1+2\gamma}\, dt.\end{aligned} \tag{3.10}$$

Now, since $\gamma = 1/4$ for $N = 3$, the inequality (3.10) with (3.6) implies the inequality

$$\begin{aligned}\int_0^\omega &\|\nabla u_{nt}\|^2\, dt\\ &\le C(M_1) + C(M_0, M)\int_0^\omega \|u_{nt}\|^{1/2}\|\nabla u_{nt}\|^{3/2}\, dt\\ &\le C(M_1) + C(M_0, M)\left(\int_0^\omega \|\nabla u_{nt}\|\, dt\right)^{1/4}\left(\int_0^\omega \|\nabla u_{nt}\|^2\, dt\right)^{3/4}.\end{aligned} \tag{3.11}$$

As for the second term of the right-hand side in the above inequality,

$$\int_0^\omega \|\nabla u_{nt}\|\,dt \le \omega^{1/2}\left(\int_0^\omega \|\nabla u_{nt}\|^2\,dt\right)^{1/2}$$

is satisfied. Thereby, the estimate (3.11) gives

$$\int_0^\omega \|\nabla u_{nt}\|^2\,dt \le C(M_0, M_1, M)\left\{\int_0^\omega \|\nabla u_{nt}\|^2\,dt\right\}^{7/8},$$

which implies the boundedness

$$\int_0^\omega \|\nabla u_{nt}\|^2\,dt \le C(M_0, M_1, M) \qquad (N = 3). \tag{3.12}$$

On the other hand, for the reason that $\gamma = 1/2$ for $N = 4$ and $\|A^\gamma u_n\| = \|\nabla u_n\|$, the inequality (3.10) gives

$$\int_0^\omega \|\nabla u_{nt}\|^2\,dt \le C(M_1) + 2C_2C_3C_4C(M)\int_0^\omega \|\nabla u_{nt}\|^2\,dt. \tag{3.13}$$

In the above inequality, seeing that $2C_2C_3C_4C(M) < 1$, we get the estimate (3.12) for $N = 4$ also. Hence, similarly, there exists $t^* \in [0, \omega]$ such that

$$\|u_{nt}(t^*)\|^2 \le \mu^{-1}\|\nabla u_{nt}(t^*)\|^2 \le \frac{1}{\mu\omega}C(M_0, M_1, M).$$

Consequently, integrating (3.9) from t^* to $t + \omega$ ($t \in [0, \omega]$) we find

$$\sup_t \|u_{nt}(t)\| \le C(M_0, M_1, M). \tag{3.14}$$

This completes the proof of Lemma 3.1.

Furthermore, we will show the following lemma.

LEMMA 3.2. *Let $u_n(t)$ be the approximate solutions given above. Then, we have*

$$\sup_t \|Au_n(t)\| \le C(M_0, M_1, M), \tag{3.15}$$

$$\sup_t \|\nabla u_{nt}(t)\| \le C(M_0, M_1, M), \tag{3.16}$$

$$\int_0^\omega \|Au_{nt}\|^2\,dt \le C(M_0, M_1, M), \tag{3.17}$$

$$\int_0^\omega \|u_{ntt}\|^2\,dt \le C(M_0, M_1, M). \tag{3.18}$$

Proof. Similarly, from (2.1)–(2.2) we get

$$(u_{nt} + Au_n, Au_n) = (f - Pu_n \cdot \nabla u_n, Au_n),$$

and

$$\|Au_n\|^2 \le \|u_{nt}\| \, \|Au_n\| + \|f\| \, \|Au_n\| + \|Pu_n \cdot \nabla u_n\| \, \|Au_n\|.$$

Applying (1.8) in Proposition 1.1 and (3.1) we see

$$\|Pu_n \cdot \nabla u_n\| \le C_1 \|A^\gamma u_n\| \, \|Au_n\| \le C_1 C(M) \|Au_n\|.$$

From this, we find also

$$\|Au_n(t)\| \le C(M_0, M_1, M).$$

Moreover, by differentiating Eq. (2.1) and making the scalar product in H_σ with Au_{nt} we see

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla u_{nt}\|^2 + \|Au_{nt}\|^2 \\ &\qquad \le \|f_t\| \, \|Au_{nt}\| + \|u_{nt}\|_{L_{2N/(N-2)}(\Omega)} \|\nabla u_n\|_{L_N(\Omega)} \|Au_{nt}\| \\ &\qquad\quad + \|u_n\|_{L_{2N/(N-2)}(\Omega)} \|\nabla u_{nt}\|_{L_N(\Omega)} \|Au_{nt}\|. \end{aligned}$$

Therefore, by using (1.10) and (1.12) we see

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla u_{nt}\|^2 + \|Au_{nt}\|^2 \le{}& \frac{1}{2}\|f_t\|^2 + \frac{1}{2}\|Au_{nt}\|^2 \\ &+ C_2 C_3 \|\nabla u_{nt}\| \, \|A^{\gamma+1/2} u_n\| \, \|Au_{nt}\| \\ &+ C_2 C_3 \|\nabla u_n\| \, \|A^{\gamma+1/2} u_{nt}\| \, \|Au_{nt}\|. \end{aligned} \tag{3.19}$$

As for the estimate (3.19), moreover, we can see that, in the case $N = 3$,

$$\begin{aligned} \frac{d}{dt}\|\nabla u_{nt}\|^2 + \|Au_{nt}\|^2 \le{}& C(M_1) \\ &+ C(M_0, M_1, M)\|\nabla u_{nt}\| \, \|Au_{nt}\| \\ &+ C(M_0, M)\|\nabla u_{nt}\|^{1/2} \|Au_{nt}\|^{3/2}, \end{aligned} \tag{3.20}$$

where we used the interpolation inequality (3.8); and in the case $N = 4$,

$$\begin{aligned} \frac{d}{dt}\|\nabla u_{nt}\|^2 + \|Au_{nt}\|^2 \le{}& C(M_1) \\ &+ C(M_0, M_1, M)\|\nabla u_{nt}\| \, \|Au_{nt}\| \\ &+ 2C_2 C_3 C(M) \|Au_{nt}\|^2. \end{aligned} \tag{3.21}$$

By applying the Young inequality $ab \leq a^p/p + b^q/q$ $(1/p + 1/q = 1)$ to the right-hand side of (3.20) and noticing $2C_2C_3C(M) < 1$ in (3.21), we obtain the desired estimate (3.17). Thus, by a similar discussion we also get the boundedness (3.16).

Lastly, we get the following equation from Eqs. (2.1)–(2.2):

$$(u_{ntt} + Au_{nt}, u_{ntt}) = (f_t - Pu_{nt} \cdot \nabla u_n - Pu_n \cdot \nabla u_{nt}, u_{ntt}).$$

As for the nonlinear term of this equation we obtain

$$|(Pu_{nt} \cdot \nabla u_n, u_{ntt})| \leq C(M_0, M_1, M)\|\nabla u_{nt}\| \, \|u_{ntt}\|,$$

$$|(Pu_n \cdot \nabla u_{nt}, u_{ntt})| \leq C(M_0, M_1, M)\|Au_{nt}\| \, \|u_{ntt}\|,$$

where we used the boundedness of $\|Au_n(t)\|$ $(n = 1, 2, \ldots)$. Thus, we can see (3.18). Consequently, the proof of Lemma 3.2 is complete.

4. PROOF OF THEOREMS 1.1–1.2

Firstly, we shall show the convergence of the approximate solutions $u_n(t)$ obtained above. Since the estimates in Lemma 2.1, Lemma 3.1, and Lemma 3.2 are valid, standard compactness arguments imply that there exists a subsequence $u_n(t)$ tending to a function $u(t)$ in such a way

$$u_n \to u \text{ weakly* in } L_\infty(\omega; D(A)) \tag{4.1}$$

$$u_n \to u \text{ strongly in } L_\infty(\omega; D(A^{1/2})) \tag{4.2}$$

$$u_{nt} \to u_t \text{ weakly* in } L_\infty(\omega; D(A^{1/2})) \tag{4.3}$$

$$u_{nt} \to u_t \text{ strongly in } L_\infty(\omega; H_\sigma), \tag{4.4}$$

and the function $u(t)$ satisfies

$$u \in H^2(\omega; H_\sigma) \cap H^1(\omega; D(A)) \cap L_\infty(\omega; D(A)) \cap W^{1,\infty}(\omega; H^1_{0,\sigma}).$$

Here, (4.1)–(4.3) are evident, and hence it is sufficient to show the convergence (4.4). In fact, the sequence $|(u_{nt}(t), w_i)|$ $(n = i, i+1, i+2, \ldots)$ is uniformly bounded and equicontinuous,

$$\left|(u_{nt}(t+h) - u_{nt}(t), w_i)\right| \leq C(M_0, M_1, M)|h|^{1/2}\|w_i\|,$$

where the w_i $(i = 1, 2, 3, \ldots)$ is the completely orthonormal system in H_σ consisting of the eigenfunctions of A as mentioned above. Therefore, using the diagonal process we can finally select a subsequence $u_{nt}(t)$ such that $u_{nt}(t)$ converges weakly and uniformly in $t \in [0, \omega]$ to an element in H_σ. Furthermore, considering the boundedness (3.16) in Lemma 3.2 we obtain the convergence (4.4).

Next, considering the above lemmas we see that $Pu \cdot \nabla u$ is well defined, and

$$\begin{aligned}
&\|Pu_n \cdot \nabla u_n - Pu \cdot \nabla u\| \\
&\qquad \le \|P(u_n - u) \cdot \nabla u_n\| + \|Pu \cdot \nabla(u_n - u)\| \\
&\qquad \le C_1\left\{\|A^{\gamma+1/4}(u_n - u)\|\,\|A^{3/4}u_n\| + \|A^{\gamma+1/4}u\|\,\|A^{3/4}(u_n - u)\|\right\} \\
&\qquad \le C(M_0, M)\left\{\|u_n - u\|^{3/4-\gamma} + \|u_n - u\|^{1/4}\right\} \\
&\qquad \to 0, \qquad \text{as } n \to \infty, \text{ uniformly in } t,
\end{aligned}$$

where we used (1.8) and (3.8). Consequently, we see that

$$(u_t + Au + Pu \cdot \nabla u, w_i) = (f, w_i), \qquad i = 1, 2, 3, \dots, \quad t \in (-\infty, +\infty). \tag{4.5}$$

We also find that this relation is valid for any $w \in H_\sigma$, owing to the estimates obtained in the previous sections. Since $Pf = f$ for any $f \in H_\sigma$, we get (1.6)–(1.7),

$$\begin{aligned}
&u_t + Au + Pu \cdot \nabla u = f \qquad \text{for any } t \in R^1, \\
&u(t + \omega) = u(t).
\end{aligned}$$

Thus, the proof of Theorem 1.1 is complete.

Lastly, we shall show Theorem 1.2 under the same assumption of Theorem 1.1. Let u and v be the solutions of the problem (1.6)–(1.7). We put $w = u - v$. Then, it follows

$$\frac{dw}{dt} + Aw + P(u - v) \cdot \nabla u + Pv \cdot \nabla(u - v) = 0,$$

and

$$\left(\frac{dw}{dt} + Aw, w\right) = -(Pw \cdot \nabla u, w),$$

because of $(Pv \cdot \nabla w, w) = 0$. Hence, it follows

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 \\
&\qquad = (Pw \cdot \nabla w, u) = (A^{-\gamma} Pw \cdot \nabla w, A^{\gamma} u) \\
&\qquad \le C_1 \|\nabla w\|^2 \|A^{\gamma} u\| \le C_1 C(M) \|\nabla w\|^2
\end{aligned} \tag{4.6}$$

by using (1.8) and (3.1). Since $C_1 C(M) < 1$, it follows

$$\frac{d}{dt}\|w\|^2 \le 2\big(C_1 C(M) - 1\big)\mu \|w\|^2 = -L\|w\|^2,$$

where $L \equiv 2(1 - C_1 C(M))\mu > 0$. Hence, it follows that

$$\|w(t)\|^2 \le \|w(0)\|^2 \exp(-Lt) \qquad \text{for any } t \in (0, \infty).$$

Since $w(t)$ is periodic in t, for any $t \in (-\infty, +\infty)$ there exists a positive integer n_0 such that $t + n_0\omega > 0$ and $\|w(t)\|^2 = \|w(t + n_0\omega)\|^2$. Hence, it follows

$$\|w(t)\|^2 \le \|w(0)\|^2 \exp(-Ln\omega) \qquad (n \ge n_0),$$

which implies $\|w(t)\| \equiv 0$. The proof of Theorem 1.2 is complete.

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